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INTENSIONAL LOGICS WITHOUT ITERATIVE AXIOMS*

I. INTRODUCTION

Among the formulas that have been used as axioms for various intensional logics, some are *iterative*: intensional operators occur within the scopes of intensional operators. Others are *non-iterative*. Some examples from modal, deontic, and tense logics are as follows.

Iterative axioms	*Non-iterative axioms*
$\Diamond p \supset \Box\Diamond p$	$\Box p \supset p$
$(p \dashv q) \supset (\Box p \dashv \Box q)$	$(p \dashv q) \supset (\Box p \supset \Box q)$
$\bigcirc(\bigcirc p \supset p)$	$-(\bigcirc p \;\&\; \bigcirc - p)$
$\mathsf{F}p \supset - \mathsf{P} - \mathsf{F}p$	$\mathsf{F}(p \supset p)$

It may be that there is no way to axiomatize an intensional logic without recourse to one or more iterative axioms; then I call that logic *iterative*. But many familiar logics can be axiomatized by means of non-iterative axioms alone; these logics I call *non-iterative*.

A *frame*, roughly speaking, is a partial interpretation for an intensional language. It provides a set I, all subsets of which are to be regarded as eligible propositional values or truth sets for formulas. It also provides, for each intensional operator in the language, a function specifying how the propositional values of formulas compounded by means of that operator are to depend on the propositional values of their immediate constituents. A frame plus an assignment of values to propositional variables are enough to yield an assignment of propositional values to all formulas in the language. Iff every interpretation based on a certain frame assigns the propositional value I (truth everywhere) to a certain formula, then we say that the formula is *valid* in the frame.

There are two different natural ways in which an intensional logic L may be said to 'correspond' to a class C of frames. For any class C of frames, there is a logic L that has as its theorems exactly those formulas that are valid in all frames in the class C. We say then that the class C

Journal of Philosophical Logic 3 (1974) 457–466.

determines the logic L. Also, for any logic L there is a class C of exactly those frames that validate all theorems of L; I shall say then that C is the *range* of L.

Any class that determines a logic L must be, of course, a subclass of the range of L. A logic is usually determined by many different subclasses of its range. But note that if any class whatever determines a logic L, then in particular the range of L determines L. (If a formula is valid in all frames in the range then it is valid in all frames in the determining subclass, so it is a theorem of L.) We call a logic *complete* iff some class of frames determines it; equivalently, iff its range determines it. We know that not every logic is complete.[1] If L is incomplete, then the range of L determines not L itself but some stronger logic L'. (Then also this one class is the range of the two different logics L and L'.) One would like to know what sorts of logics can be incomplete. A partial answer is: iterative ones only. *Every non-iterative logic is complete.* Somewhat more precisely: every non-iterative classical propositional intensional logic is complete with respect to classical frames with unrestricted valuations.

The proof is an adaptation of the method of filtrations. As a bonus, it yields a decidability result and an easy route to more specialized completeness results for non-iterative logics.

II. LANGUAGE

Consider a formal language partially specified as follows. (1) There are countably many propositional variables $p, q, r, \ldots$; and (2) propositional variables are formulas. (3) There are enough of the truth-functional connectives: & and $-$, let us say. (The rest may be regarded as metalinguistic abbreviations.) (4) There are at most countably many intensional operators, each of which is an n-ary connective for some number n, $n \geqslant 0$. (5) The result of applying any n-ary truth-functional connective or intensional operator to any n formulas (in appropriate order and with appropriate punctuation) is a formula. (6) There are no other formulas.

A formula is *iterative* iff it has a subformula of the form $O(\emptyset_1 \ldots \emptyset_n)$ where O is an intensional operator and at least one of the formulas $\emptyset_1 \ldots \emptyset_n$ already contains an occurrence of some intensional operator. Note that I, unlike some authors, am *not* excluding iterative formulas from the language itself. They are perfectly well-formed; and the logics we shall consider will all have iterative formulas among their theorems. It is only as axioms that iterative formulas will be forbidden.

Since there are only countably many formulas, we may suppose given some arbitrary fixed enumeration of them; thus we may legitimately refer to the *first* formula of a given sort.

III. LOGICS

We identify a logic with the set of its theorems. A set of formulas is a *logic* (more fully, a *classical propositional intensional logic*) iff it is closed under these three rules (taking 'theorem' as 'member of the set'):

TF. If ϕ follows by truth-functional logic from zero or more theorems, then ϕ is a theorem.

Substitution. If ϕ is the result of uniformly substituting formulas for propositional variables in a theorem, then ϕ is a theorem.

Interchange. If $\psi \equiv \chi$ is a theorem, and if ϕ' is the result of substituting ψ for χ at one or more places in ϕ, then $\phi \equiv \phi'$ is a theorem.

A set of formulas A is an *axiom set for* a logic L iff L is the least logic that includes all members of A as theorems. A logic L is *iterative* iff every axiom set for L contains at least one iterative formula; L is *non-iterative* iff at least one axiom set for L contains no iterative formula. A logic L is *finitely axiomatizable* iff some axiom set for L is finite.

IV. FRAMES, INTERPRETATIONS, VALIDITY

A *frame* (or more fully, a *classical frame with unrestricted valuations*) is any pair $\langle I, f \rangle$ of a set I and a family f of functions, indexed by the intensional operators of the language, such that if 0 is an n-ary operator then f_0 is a function from n-tuples of subsets of I to subsets of I.

An *interpretation* is any function $[\![\ ]\!]$ which assigns to each formula ϕ of the language a set $[\![\phi]\!]$. (We may think of $[\![\phi]\!]$ as the truth set or propositional value of ϕ under the interpretation.) An interpretation $[\![\ ]\!]$ is *based on* a frame $\langle I, f \rangle$ iff the following conditions hold for all formulas and operators:

(0) $[\![\phi]\!] \subseteq I$,

(1) $[\![\phi \,\&\, \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$,

(2) $[\![-\emptyset]\!] = I - [\![\emptyset]\!]$,
(3) $[\![O(\emptyset_1 \ldots \emptyset_n)]\!] = f_O([\![\emptyset_1]\!] \ldots [\![\emptyset_n]\!])$.

A formula $\emptyset$ is *valid in* a frame $\langle I, f\rangle$ iff, for every interpretation $[\![\ ]\!]$ based on $\langle I, f\rangle$, $[\![\emptyset]\!] = I$. If C is a class of frames and L is a logic (in the precise senses we have now defined), we recall that C *determines* L iff the theorems of L are all and only the formulas valid in all frames in the class C; and that C is the *range* of L iff C contains all and only those frames in which all theorems of L are valid. (We can also verify at this point that every class C of frames determines a logic; that is, that the set of formulas valid in all frames in C is closed under *TF*, *Substitution*, and *Interchange*.)

We shall need the following two lemmas; I omit the obvious proofs.

LEMMA 1. Suppose that A is an axiom set for a logic L, and that all formulas in A are valid in a frame $\langle I, f\rangle$. Then all theorems of L are valid in $\langle I, f\rangle$ (which is to say that $\langle I, f\rangle$ belongs to the range of L).

LEMMA 2. Suppose that $\emptyset'$ is the result of substituting formulas ψ_p, ψ_q, ... uniformly for the propositional variables $p, q, \ldots$ in a formula $\emptyset$. Suppose that $[\![\ ]\!]$ and $[\![\ ]\!]'$ are two interpretations based on the same frame such that $[\![\psi_p]\!] = [\![p]\!]'$, $[\![\psi_q]\!] = [\![q]\!]'$, and so on. Then $[\![\emptyset']\!] = [\![\emptyset]\!]'$.

V. FILTRATIONS

Let L be any logic. A set of sentences is *L-consistent* iff no theorem of L is the negation of any conjunction of sentences in the set. A *maximal L-consistent* set is an L-consistent set that is not properly included in any other one. For any logic L, we have these familiar lemmas.

LEMMA 3. Any L-consistent set is included in a maximal L-consistent set.

LEMMA 4. If i is any maximal L-consistent set, then (1) a conjunction $\emptyset \,\&\, \psi$ belongs to i iff both conjuncts $\emptyset$ and ψ belong to i; (2) a negation $-\emptyset$ belongs to i iff $\emptyset$ does not belong to i; (3) a biconditional $\emptyset \equiv \psi$ belongs to i iff both or neither of $\emptyset$ and ψ belong to i.

LEMMA 5. Any theorem of L belongs to all maximal L-consistent sets.

Next, let θ be any formula. A *θ-description* is any set of zero or more subformulas of θ together with the negations of all other subformulas of θ. For every L-consistent θ-description D, let i_D be an arbitrarily chosen one of the maximal L-consistent supersets of D; and let I be the set of all these i_D's. I contains one member for each L-consistent θ-description. But there are only 2^s θ-descriptions in all, where s is the number of subformulas of θ. Wherefore we have:

LEMMA 6. I is finite, and an upper bound on its size can be computed by examination of θ.

A theorem of L must belong to all maximal L-consistent sets, and hence to all members of I. We have a restricted converse:

LEMMA 7. Suppose that $\emptyset$ is a truth-functional compound of subformulas of θ. Then $\emptyset$ belongs to all members of I iff $\emptyset$ is a theorem of L.

Proof: If D is a θ-description and $\emptyset$ is a truth-functional compound of subformulas of θ, then $D \cup \{\emptyset\}$ and $D \cup \{-\emptyset\}$ cannot both be consistent in truth-functional logic, and so cannot both be L-consistent. *Case 1*: there is some θ-description D such that $D \cup \{-\emptyset\}$ is L-consistent. Then $\emptyset$ is not a theorem of L. Then also $D \cup \{\emptyset\}$ is not L-consistent, and hence $\emptyset$ does not belong to i_D. *Case 2*: there is no θ-description D such that $D \cup \{-\emptyset\}$ is L-consistent. Then $\emptyset$ is a theorem of L, by the rule *TF*. Then also $\emptyset$ must belong to every i_D in I. Q.E.D.

It will be convenient to introduce the following notation: let $/\emptyset/ =df\{i \in I : \emptyset \in i\}$.

We next specify an assignment of formulas, called *labels*, to all members and subsets of I. Any member of I is i_D for some one L-consistent θ-description D; let the label of i_D be the first conjunction (first in our arbitrary enumeration of all formulas) of all formulas in D. Any non-empty subset of I contains finitely many members; let the label of the subset be the first disjunction of the labels of all of its members. Finally, let the label of the empty set be the contradiction $\theta \,\&\, -\theta$. Our assignment of labels has the following properties.

LEMMA 8. Every subset of I has as its label a truth-functional compound of subformulas of θ.

LEMMA 9. If $\emptyset$ is the label of a subset J of I, then $/\emptyset/=J$.

We next specify a family f of functions such that $\langle I,f\rangle$ is a frame. Let 0 be any n-ary operator, let $J_1...J_n$ be any n subsets of I, and let $\emptyset_1...\emptyset_n$ be the labels of $J_1...J_n$ respectively; then $f_0(J_1...J_n)$ is to be $/0(\emptyset_1...\emptyset_n)/$.

Next, let $[\![\ ]\!]$ be the interpretation based on the frame $\langle I,f\rangle$ such that the following equation (*) holds whenever the formula $\emptyset$ is a propositional variable:

(*) $[\![\emptyset]\!]=/\emptyset/$.

We call the pair of the specified frame $\langle i,f\rangle$ and the specified interpretation $[\![\ ]\!]$ based on that frame a *filtration* (*for* the logic L and the formula θ).

Filtrations are of interest because the equation (*) carries over from propositional variables to certain compound formulas, in accordance with the following two lemmas.

LEMMA 10. (*) holds for any truth-functional compound of constituent formulas for which (*) holds.

The proof is immediate. We need only consider conjunctions and negations, and compare clauses (1) and (2) of the definition of an interpretation based on a frame with parts (1) and (2) of Lemma 4. We turn next to compounding by means of the intensional operators.

LEMMA 11. (*) holds for a formula of the form $0(\emptyset_1...\emptyset_n)$ whenever (a) (*) holds for each of the constituent formulas $\emptyset_1...\emptyset_n$, and (b) $\emptyset_1...\emptyset_n$ are all truth-functional compounds of subformulas of θ.

Proof: Let $\psi_1...\psi_n$ be the labels of the sets $[\![\emptyset_1]\!]...[\![\emptyset_n]\!]$ respectively. By definition of f, $[\![0(\emptyset_1...\emptyset_n)]\!]=f_0([\![\emptyset_1]\!]...[\![\emptyset_n]\!])=/0(\psi_1...\psi_n)/$. Also, by Lemma 9 and hypothesis (*a*), $/\psi_1/=/\emptyset_1/,...,$ and $/\psi_n/=/\emptyset_n/$. Consider the biconditionals $\psi_1\equiv\emptyset_1,...,\psi_n\equiv\emptyset_n$. By part (3) of Lemma 4, they belong to all members of I; by hypothesis (*b*) and Lemma 8 they are truth-functional compounds of subformulas of θ; so it follows by Lemma 7 that they are theorems of L. By repeated use of Interchange, $0(\psi_1...\psi_n)\equiv$ $\equiv 0(\emptyset_1...\emptyset_n)$ also is a theorem of L; wherefore by Lemma 7 and part (3) of Lemma 4, $/0(\psi_1...\psi_n)/=/0(\emptyset_1...\emptyset_n)/$. Q.E.D.

The corollaries which one usually aims at in working with filtrations are these:

LEMMA 12. (*) holds for all subformulas of θ.

LEMMA 13. (*) holds for θ itself.

But our lemmas on truth-functional and intensional compounding overshoot the subformulas of θ, and this overshoot is crucial for our present task. We have this further corollary:

LEMMA 14. Let $\emptyset$ be a non-iterative formula, let $\psi_p, \ldots \psi_q, \ldots$ be truth-functional compounds of subformulas of θ, and let $\emptyset'$ be the result of substituting ψ_p, $\psi_q, \ldots$ uniformly for the propositional variables p, $q, \ldots$ in $\emptyset$. Then (*) holds for $\emptyset'$.

Lemma 14 will apply, in particular, when $\emptyset$ is a non-iterative axiom and ψ_p, $\psi_q, \ldots$ are labels of various subsets of I.

LEMMA 15. Suppose that $\emptyset$ is a non-iterative formula, and that $\emptyset$ belongs to some axiom set for the logic L. Then $\emptyset$ is valid in $\langle I, f \rangle$.

Proof: We must show that for every interpretation $[\![\ ]\!]'$ based on $\langle I, f \rangle$, $[\![\emptyset]\!]' = I$. Given any such $[\![\ ]\!]'$, let $\emptyset'$ be the result of substituting the label ψ_p of the set $[\![p]\!]'$ uniformly for the propositional variable p, the label ψ_q of $[\![q]\!]'$ uniformly for $q, \ldots$, in $\emptyset$. Since (*) holds for each of these labels, we have $[\![\psi_p]\!] = /\psi_p/ = [\![p]\!]'$, $[\![\psi_q]\!] = /\psi_q/ = [\![q]\!]'$, and so on; so by Lemma 2, $[\![\emptyset]\!]' = [\![\emptyset']\!]$. By Lemma 14, $[\![\emptyset']\!] = /\emptyset'/$. Since $\emptyset$ is an axiom and hence a theorem of L, it follows by Substitution that $\emptyset'$ also is a theorem of L, so $/\emptyset'/ = I$. Q.E.D.

LEMMA 16. If L is a non-iterative logic, then $\langle I, f \rangle$ is in the range of L.

Proof: Let A be an axiom set for L that contains no iterative formulas. By Lemma 15, all formulas in A are valid in $\langle I, f \rangle$, so by Lemma 1 all theorems of L are valid in $\langle I, f \rangle$. Q.E.D.

LEMMA 17. If θ is not a theorem of L, θ is invalid in $\langle I, f \rangle$.

Proof: If θ is not a theorem of L, then by Lemmas 7 and 13 $[\![\theta]\!] \neq I$. Q.E.D.

We can summarize our lemmas on filtrations as follows.

THEOREM 1. If L is any non-iterative logic and θ is any non-theorem of L, then there is a frame $\langle I, f \rangle$, belonging to the range of L, in which θ is invalid. Further, I is finite and an upper bound on its size can be computed by examination of θ.

VI. GENERAL COMPLETENESS AND DECIDABILITY OF NON-ITERATIVE LOGICS

Our general completeness theorem follows at once.

THEOREM 2. *Every non-iterative logic is complete.*

As for decidability, it suffices to note that if a logic L is finitely axiomatizable, then it is a decidable question whether or not a given formula θ is valid in all frames $\langle I, f \rangle$ in the range of L such that the size of I does not exceed a bound computed by examination of θ. Although there are infinitely many such frames (except in trivial cases), we may ignore irrelevant differences among them: differences between isomorphic frames, or between frames that differ only with respect to operators that do not appear in θ. It is enough to search through finitely many representative cases, all of them decidable. Therefore we have:

THEOREM 3. Every finitely axiomatizable non-iterative logic is decidable.

VII. SPECIAL COMPLETENESS RESULTS

We often wish to establish not only that a logic is complete but also that it is determined by some subclass of its range that is of special interest to us. If we have to do with non-iterative logics, Theorem 1 facilitates the proof of special completeness results by allowing us to ignore problems that arise only in infinite cases. To show that a class C determines a non-iterative logic L, it is enough to show that C is intermediate in inclusiveness between the full range of L and the part of the range consisting of finite frames; in other words, to show (1) that C is a subclass of the range of L, and (2) that $\langle I, f\rangle$ belongs to C whenever $\langle I, f\rangle$ is in the range of L and I is finite.

By way of example, consider the non-iterative regular modal logics. Let our language contain a single, unary operator $\Box$; and let L be a logic having as a theorem

C. $\qquad \Box(p \,\&\, q) \equiv (\Box p \,\&\, \Box q).$

Any such logic is called a *regular modal logic*. Further, let L be non-iterative. The frames of special interest to us will be those that derive from the standard accessibility semantics for modal logic. A *modal structure* is a triple $\langle I, N, R\rangle$ of a set I, a subset N of I (the *normal* set), and a relation R (the *accessibility* relation) included in $N \times I$. A frame $\langle I, f\rangle$ *corresponds* to such a modal structure iff, for any subset A of I, $f_\Box A = \{i \in N$: whenever $iRj, j \in A\}$. That gives the standard semantic rule: if an interpretation $[\![\ ]\!]$ is based on a frame corresponding to a modal structure, then always i belongs to $[\![\Box\phi]\!]$ iff i is normal and whenever j is accessible from i, j belongs to $[\![\phi]\!]$. The correspondence thus defined is one-one between all modal structures and exactly those frames $\langle I, f\rangle$ that obey the following distributive condition: whenever $\mathscr{A}$ is a set of subsets of I, then $f_\Box(\cap\mathscr{A}) = \bigcap\{f_\Box A : A \in \mathscr{A}\}$. (Let $\langle I, f\rangle$ be such a frame; then the corresponding modal structure is obtained by taking $N = f_\Box I$, iRj iff i belongs to N and whenever i belongs to $f_\Box A$, j belongs to A.) Not every frame in the range of L satisfies this distributive condition; to validate **C**, it is enough that the distributive condition should hold for finite $\mathscr{A}$. But when the frame itself is finite, the case of infinite $\mathscr{A}$ does not arise. Thus if $\langle I, f\rangle$ is in the range of L and I is finite, then $\langle I, f\rangle$ satisfies the distributive condition and corresponds to a modal structure. The subclass of the range of our non-iterative regular modal logic L comprising those frames that correspond to modal structures is therefore a determining class for L. From here we can go instantly to the standard completeness results for such logics as **C2**, **D2**, **E2**, **K**, **D**, and **T**.

For another example, consider the non-iterative ones of the V-logics discussed in my *Counterfactuals* (Blackwell, 1973). Let our language contain a single, binary operator $\preccurlyeq$; and let L be a logic having as theorems

Trans. $(p \preccurlyeq q \; \& \; q \preccurlyeq r) \supset p \preccurlyeq r$,
Connex. $p \preccurlyeq q \vee q \preccurlyeq p$,
Dis. $((p \vee q) \preccurlyeq r) \equiv (p \preccurlyeq r \vee q \preccurlyeq r)$.

I call any such logic a *V-logic*. (My Rule for Comparative Possibility in *Counterfactuals* is interderivable with the combination used here of **Dis** and the rule of Interchange.) Further, let L be non-iterative. The frames $\langle I, f \rangle$ of special interest to us will be those derived from families of rankings of I. A *ranking structure* $\langle K_i, R_i \rangle_{i \in I}$ is an assignment to each i in I of a subset K_i of I (the *evaluable* set) and a weak ordering R_i of K_i (the *ranking*). The frame $\langle I, f \rangle$ *corresponds* to such a ranking structure iff, for any subsets A and B of I, $f_{\preccurlyeq}(A, B) = \{i \in I$: for every k in $B \cap K_i$ there is some j in $A \cap K_i$ such that $jR_ik\}$. That gives the semantic rule: if an interpretation $[\![\;]\!]$ is based on a frame corresponding to a ranking structure, then always i belongs to $[\![\phi \preccurlyeq \psi]\!]$ iff, relative to i, for every evaluable k in $[\![\psi]\!]$, some evaluable j in $[\![\phi]\!]$ is ranked at least as highly as k is. A frame corresponds to every ranking structure, but not conversely; however, if $\langle I, f \rangle$ is a frame in the range of L and I is finite, then $\langle I, f \rangle$ corresponds to the ranking structure $\langle K_i, R_i \rangle_{i \in I}$ obtained as follows. For any i in I, let K_i be $\{j \in I: i \notin f_{\preccurlyeq}(\Lambda, \{j\})\}$; for any i in I and j, k in K_i, let jR_ik iff i belongs to $f_{\preccurlyeq}(\{j\}, \{k\})$. The subclass of the range of our non-iterative V-logic L comprising those frames that correspond to ranking structures is therefore a determining class for L. From here we can instantly reach the completeness results given in *Counterfactuals* for those V-logics that can be axiomatized without the non-iterative axioms **U** and **A**.

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NOTES

* I am grateful to Peter Gärdenfors for valuable discussion of the subject of this paper.

[1] See Kit Fine, 'An Incomplete Logic Containing S4' and S. K. Thomason, 'An Incompleteness Theorem in Modal Logic', both forthcoming in *Theoria*; also M. Gerson, 'The Inadequacy of the Neighborhood Semantics for Modal Logic', unpublished. It should, however, be noted that the situation changes if we think of a frame as providing a domain of propositional values which may *not* be the full power set of any set I. Under this changed concept of a frame, all classical intensional logics are complete; see Bengt Hansson and Peter Gärdenfors, 'A Guide to Intensional Semantics', in *Modality, Morality and Other Problems of Sense and Nonsense: Essays Dedicated to Sören Halldén* (Lund, 1973). Still broader general completeness results of the same sort have been obtained by Richard Routley and Robert K. Meyer.