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HEMPEL AND OPPENHEIM ON EXPLANATION*

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Hempel and Oppenheim, in their paper 'The Logic of Explanation', have offered an analysis of the notion of scientific explanation. The present paper advances considerations in the light of which their analysis seems inadequate. In particular, several theorems are proved with roughly the following content: between almost any theory and almost any singular sentence, certain relations of explainability hold.

Hempel and Oppenheim have offered in Part III of [3] an analysis of the notion of scientific explanation. It is the purpose of the present paper to advance considerations in the light of which their analysis seems to us inadequate.

Like Hempel and Oppenheim, we consider a formal language L which has the syntactical structure of the lower predicate calculus without identity. The logical constants of L are $\sim$, $\cdot$, v, $\supset$, $\equiv$, ( ), $\exists$, which are the respective symbols of negation, conjunction, disjunction, implication, equivalence, universal quantification, and existential quantification. The vocabulary of L contains in addition parentheses, individual variables, individual constants, and predicates of any desired finite degree. *Formulas of L* (which may contain free variables) and *sentences of L* (which may not contain free variables) are characterized in the usual way. Throughout the following we shall have the fixed language L in mind and mean by 'formula' and 'sentence', 'formula of L' and 'sentence of L' respectively. We shall also suppose that an interpretation for L has been fixed, so that we may meaningfully speak of *true* and *false* sentences.[2]

Logical provability, *logical derivability* (of a sentence from a set of sentences or from another sentence), and *logical equivalence* are to be understood in the manner usual within the lower predicate calculus without identity. We use the notations '$\vdash S$' and '$K \vdash S$' for logical provability and logical derivability respectively. For the convenience of the reader, the relevant notions of the Hempel and Oppenheim paper are defined here. A number of ancillary notions have been omitted, and, consequently, definitional chains compressed; but the reader will find the following definitions immediate consequences of those given by Hempel and Oppenheim. There is one exception, noted in footnote 3, which corresponds to a minor correction introduced in order to remain faithful to what we regard as the authors' intention.

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[2] The interpretation for L is subjected by Hempel and Oppenheim to certain special requirements that are here irrelevant.

A *singular* sentence is one in which no variables occur, an *atomic* sentence one which consists of a predicate followed by individual constants, and a *basic* sentence either an atomic sentence or the negation of an atomic sentence.

A *fundamental law* is a true sentence consisting of one or more universal quantifiers followed by an expression without quantifiers or individual constants. A sentence S is called a *derivative law* if (1) S consists of one or more universal quantifiers followed by an expression without quantifiers, (2) S is not logically equivalent to any singular sentence, (3) at least one individual constant occurs in S, and (4) there is a class K of fundamental laws such that S is logically derivable from K. A *law* is a sentence which is either a fundamental law or a derivative law.

A *fundamental theory* is a true sentence consisting of one or more quantifiers followed by an expression without quantifiers or individual constants. A sentence S is called a *derivative theory* if (1) S consists of one or more quantifiers followed by an expression without quantifiers, (2) S is not logically equivalent to any singular sentence, (3) at least one individual constant occurs in S, and (4) there is a class K of fundamental theories such that S is logically derivable from K. A *theory* is a sentence which is either a fundamental theory or a derivative theory.[3]

An ordered couple (T, C) of sentences is an *explanans* for a singular sentence E if and only if the following conditions are satisfied:

(1) T is a theory,
(2) T is not logically equivalent to any singular sentence,
(3) C is singular and true,
(4) E is logically derivable from the set $\{T, C\}$, and
(5) there is a class K of basic sentences such that C is logically derivable from K, and neither E nor $\sim T$ is logically derivable from K.

A singular sentence E is *explainable* by a theory T just in case there is a singular sentence C such that the ordered couple (T, C) is an explanans for E.

The following simple example is sufficient in our opinion to indicate a divergence between the notion defined above and the customary notion of explainability. Let us assume that there are no mermaids and that the Eiffel Tower is a good conductor of heat. Then the singular sentence 'the Eiffel Tower is a good conductor of heat' (or rather, its translation in the language L) will be explainable by 'all mermaids are good conductors of heat'. To see this, choose for C in the definition of 'explainable' the singular sentence 'if the

[3] The 'minor correction' mentioned above arises in connection with the definition of a derivative theory. The authors call a sentence *essentially universal* if it consists of universal quantifiers followed by an expression without quantifiers and is not equivalent to a singular sentence. On the other hand, they call a sentence *essentially generalized* if it is simply not equivalent to a singular sentence. The intention, however, seems to be to require in addition that an essentially generalized sentence consists of quantifiers followed by an expression without quantifiers. The choice between the two possible definitions of the auxiliary notion 'essentially generalized' affects the ensuing definition of a derivative theory, but does not essentially affect the results of the present paper.

Eiffel Tower is not a mermaid, then the Eiffel Tower is a good conductor of heat', and for K in the definition of 'explanans' the class whose only member is the basic sentence 'the Eiffel Tower is a mermaid'.

There are innumerable obvious ways in which to circumvent this example. The definition of 'explainable' is, we think, more conclusively trivialized by Theorems 1-5 below, which show, roughly speaking, that according to the definition of Hempel and Oppenheim a relation of explainability holds between almost any theory and almost any true singular sentence. But first several lemmas concerning the predicate calculus must be stated. Lemma 2 is proved in [1], Lemma 3 is well known, and Lemma 4 is obvious.

Lemma 1. If S is a singular sentence and T a sentence consisting of universal quantifiers followed by an expression without quantifiers or individual constants, and

$$\vdash S \supset T,$$

then either $\vdash \sim S$ or $\vdash T$.

Proof. Assume to the contrary that neither $\vdash \sim S$ nor $\vdash T$. Then, by the Completeness Theorem of Gödel ([2]), there are models M and N (corresponding to the language L) such that S is true in M and $\sim T$ is true in N; we may in fact choose M and N so that their universes will be disjoint. Form the model P as follows: the universe of P is to be the union of the universes of M and N; to each predicate of L, P is to assign an extension which is the union of the two extensions assigned by M and N; and the individual constants of L are to designate in P what they designate in M. It is then easy to see that both S and $\sim T$, in view of their structure and the fact that they are true in M and N respectively, are true in P. But this contradicts the assumption that $\vdash S \supset T$.

Lemma 2. If S, T are sentences containing no common predicates, and $\vdash S \vee T$, then $\vdash S$ or $\vdash T$.

Lemma 3. If the sentence $S(b_1, ..., b_n)$ is obtained from the formula $S(x_1, ..., x_n)$ by replacing all free occurrences of the variables $x_1, ..., x_n$ by the individual constants $b_1, ..., b_n$, where $b_i = b_j$ if and only if $x_i = x_j$, and $\vdash S(b_1, ..., b_n)$, then also $\vdash (x_1) ... (x_n)\, S(x_1, ..., x_n)$.

Lemma 4. If S, T are singular sentences with no common atomic subsentences, and $\vdash S \vee T$, then $\vdash S$ or $\vdash T$.

Theorem 1. Let T be any fundamental law and E any singular true sentence such that neither T nor E is logically provable and T, E have no predicates in common. Assume in addition that there are at least as many individual constants in L beyond those occurring in E as there are variables in T, and at least as many one-place predicates in L beyond those occurring in T and E as there are individual constants in E. Then there is a fundamental law T' which is logically derivable from T and such that E is explainable by T'.

Proof. Let T and E have the respective forms

$(y_1) ... (y_n)\, T_0(y_1, ..., y_n)$,

$E_0(a_1, ..., a_p)$,

where $T_0(y_1, ..., y_n)$ and $E_0(a_1, ..., a_p)$ are without quantifiers, $y_1, ..., y_n$ are variables, $a_1, ..., a_p$ are distinct individual constants, and moreover are exactly the individual constants occurring in E. Let $J_1, ..., J_p$ be distinct new one-place predicates of L; such predicates exist by the hypothesis of the theorem. Let T^* be the sentence

$$T \vee (x_1) \dots (x_p) [(J_1x_1 \cdot \dots \cdot J_px_p) \supset E_0(x_1, ..., x_p)].$$

(Here $x_1, ..., x_p$ are to be distinct variables.) Clearly, there is a sentence T' such that T' consists of universal quantifiers followed by an expression without quantifiers or individual constants and T' is logically equivalent to T^*. Let $b_1, ..., b_n$ be new individual constants of L such that, for all i, j, $b_i = b_j$ if and only if $y_i = y_j$, and let C be the sentence

$$[T_0(b_1, ..., b_n) \vee \sim J_1a_1 \vee \dots \vee \sim J_pa_p] \supset E_0(a_1, ..., a_p).$$

It is clear that

(1) T' is a fundamental theory, and hence a theory,

(2) C is singular and true

(since the consequent of C is true), and

(3) $\{T', C\} \vdash E$

In order to show that

(4) T' is not logically equivalent to a singular sentence, assume to the contrary that there is some singular sentence, say S, such that $\vdash S \equiv T'$. It follows that $\vdash S \supset T'$, and therefore, by Lemma 1, that $\vdash \sim S$ or $\vdash T'$. But $\sim S$, being equivalent to $\sim T'$is false and hence not provable. Therefore T' is provable, but then so is T^*. Since the two disjuncts of T^* contain no common predicates or individual constants, we conclude by Lemma 2 that one of the two disjuncts is logically provable. But the first disjunct is simply T and hence is not logically provable. Thus we have:

$$\vdash (x_1) \dots (x_p) \; [(J_1x_1 \cdot \dots \cdot J_px_p) \supset E_0 \, (x_1, ..., x_p)]$$

But if a sentence is logically provable, so is any other sentence obtained from it by the *Substitution Rule on Predicates* (cf. [4]). Thus in the sentence above we may substitute for the formulas $J_1x_1, ..., J_px_p$ the respective formulas $J_1x_1 \vee \sim J_1x_1, ..., J_px_p \vee \sim J_px_p$, and conclude that

$$\vdash (x_1) \dots (x_p) \, \mathrm{E}_0(x_1, ..., x_p),$$

and hence $\vdash E$, which contradicts the hypothesis of the theorem.

It remains only to show that condition (5) of the definition of *explanans* is satisfied by T', C, and E. By Lemma 3 and the fact that T is not logically provable, we see that $T_0(b_1, ..., b_n)$ is not logically provable. Hence its negation is true in some model M. Let K_1 be the set of basic sentences which are true in M and which contain no predicates or individual constants beyond those occurring in $T_0(b_1, ..., b_n)$.

We can show by a simple induction on the length of S, that if S is any singular sentence which contains no predicates or individual constants beyond those occurring in $T_0(b_1, ..., b_n)$, then $K_1 \vdash S$ if and only if S is true in M. Thus, in particular,

(5) $K_1 \vdash \sim T_0(b_1, ..., b_n)$.

Let K_2 be the result of adding to the set K_1 the sentences $J_1a_1, ..., J_pa_p$. Then K_2 is a class of basic sentences; we shall show that it satisfies condition (5) of the definition of an explanans. In the first place, it is clear in view of (5) above that

(6) C is logically derivable from K_2.

Next we must show that

(7) E is not logically derivable from K_2.

Assume to the contrary that $K_2 \vdash E$, and let S be a conjunction of all the elements of K_1. Then $\vdash \sim S \vee \sim J_1a_1 \vee ... \vee \sim J_pa_p \vee E$, and hence, by several applications of Lemma 4, either $\vdash \sim S$, $\vdash E$, or $\vdash \sim J_ia_i$ for some i such that $1 \leqslant i \leqslant p$. But all of these alternatives are impossible, for the following respective reasons: S has a model M, E was assumed not to be logically provable, and no basic sentence is logically provable.

We must show next that

(8) $\sim T'$ is not logically derivable from K_2.

Let K_3 be the class consisting of the following sentences:

S,
J_ia_i (for $1 \leqslant i \leqslant p$),
$\sim J_ia_j$ (for $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant p$, $i \neq j$),
$\sim J_ib_j$ (for $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant n$),
$E_0(a_1, ..., a_p)$.

Now K_3 is consistent since it consists entirely of singular sentences and contains no truth-functional contradiction. Thus K_3 is true in some model N. In the submodel N' of N whose universe consists of the designata in N of a_1, ..., a_p, b_1, ..., b_n, all sentences in K_2 are clearly true, and so is the second disjunct of T^*. Thus T', together with all members of K_2, is true in N', and $\sim T'$ cannot be logically derivable from K_2.

By (1), (4), (2), (3), (6), (7), and (8), (T', C) is an explanans for E, and hence E is explainable by T'.

This completes the proof of the theorem.

Theorem 1 applies only to fundamental laws, but can be extended, as the next theorem asserts, to arbitrary fundamental theories which are capable of explaining anything. Again a lemma must be proved first.

Lemma 5. If T is a sentence from which some singular sentence which is not logically provable is logically derivable, then there is a singular sentence which is not logically provable, which is logically derivable from T, and which contains no predicates beyond those in T.

Proof. If $\vdash \sim T$, the conclusion is obvious. Assume, therefore, that $\sim T$ is not logically provable. Let S be a singular sentence which is logically derivable from T but not logically provable. Let S' be a conjunctive normal form of S, and let S'' be a conjunct of S' which is not logically provable; there must be such a conjunct, for otherwise S would be logically provable. Now S'' will have the form

$$S_1 \vee \ldots \vee S_n,$$

where each S_i is basic. Obtain S''' from S'' by omitting those basic disjuncts which contain predicates not occurring in T. (The sentence S'' does not entirely vanish in the process, for it must contain at least one predicate occurring in T. To see this, assume otherwise. Then, since $\vdash T \supset S''$, we can apply Lemma 2 and conclude that either $\vdash \sim T$ or $\vdash S''$. But both alternatives are excluded by our assumptions.)

Since $\vdash T \supset (S_1 \vee \ldots \vee S_n)$, we see, by the Substitution Rule on Predicates and the fact that no sentence and its negation can both occur among $S_1, \ldots, S_n$, that

$$\vdash T \supset (S_1{}^* \vee \ldots \vee S_n{}^*),$$

where, for $i = 1, \ldots, n$, $S_i{}^*$ is determined as follows. If the predicate in S_i occurs in T, $S_i{}^* = S_i$. Otherwise $S_i{}^*$ is the result of substituting a tautology or a contradiction, according as S_i is or is not a negation, for the atomic subsentence of S_i. But clearly $\vdash (S_1{}^* \vee \ldots \vee S_n{}^*) \equiv S'''$. It follows that $\{T\} \vdash S'''$. Further, S''' is not logically provable, since S'' is not; and S''' clearly contains no predicates beyond those in T.

Theorem 2. Let T be a fundamental theory and E a singular true sentence such that neither T nor E is logically provable and T, E have no predicates in common. Assume in addition that L contains infinitely many individual constants, and at least as many one-place predicates beyond those occurring in T and E as there are individual constants in E. Assume also that some singular sentence is explainable by T. Then there is a fundamental law T' which is logically derivable from T and such that E is explainable by T'.

Proof. Let E_0 be a singular sentence explainable by T. Then there is a singular sentence C such that (T, C) is an explanans for E_0. It follows that $\{T\} \vdash C \supset E_0$. Further, there is a class K of basic sentences such that $K \vdash C$, but not $K \vdash E_0$. Hence it is not the case that $C \supset E_0$ is logically provable. Thus there is a singular sentence which is logically derivable from T but not logically provable. Hence, by Lemma 5, there is such a sentence which, in addition, contains no predicates beyond those in T; let it have the form

$$S(a_1, \ldots, a_p),$$

where $a_1, \ldots, a_p$ is an exhaustive and non-repeating list of the individual constants occurring in $S(a_1, \ldots, a_p)$. Let $x_1, \ldots, x_p$ be distinct variables, and let T_0 be the sentence

$$(x_1) \ldots (x_p)\, S(x_1, \ldots, x_p).$$

By Lemma 3 and the fact that T is a sentence containing no individual constants,

$$\{T\} \vdash T_0;$$

thus T_0 is true, and hence is a fundamental law. Further, T_0 is not logically provable, since $S(a_1, \ldots, a_p)$ is not; and T_0 clearly contains no predicates beyond those occurring in T. Thus we may apply Theorem 1 and infer the existence of a fundamental law T' which is logically derivable from T_0 and such that E is explainable by T'. But then T' satisfies the conclusion of Theorem 2.

In the next two theorems several of the hypotheses of Theorems 1 and 2 are removed; the 'intermediate' law whose existence is asserted becomes, however, a derivative rather than a fundamental law.

Theorem 3. Let T be any fundamental law and E any true singular sentence such that neither T nor E is logically provable. Assume in addition that there are at least as many individual constants in L beyond those occurring in E as there are variables in T. Then there is a derivative law logically derivable from T by which E is explainable.

Proof. Case I: $\vdash T \supset E$. Let T' be a derivative law equivalent to T; by Lemma 1, there obviously are such laws. The reader can easily check to see that if C is any singular tautology, then (T', C) is an explanans for E, where the required class K is the null class.

Case II: It is not the case that $\vdash T \supset E$. Let T' be a sentence which consists of universal quantifiers followed by an expression without quantifiers, which contains the individual constants of E, and such that

(1) $\vdash T' \equiv T \vee E$.

From which it follows that

(2) $\vdash T \supset T'$.

It is required to show that

(3) T' is a derivative law,

and for this it is sufficient to show that

(4) T' is not logically equivalent to any singular sentence.

Assume to the contrary that T' is logically equivalent to the singular sentence S. Then

(5) $\vdash T \vee E \equiv S$.

Hence $\vdash (S \cdot \sim E) \supset T$, which by Lemma 1 implies that $\vdash \sim(S \cdot \sim E)$, and hence that $\vdash S \supset E$. Thus, by (5), $\vdash T \supset E$, contrary to the assumption of the present case.

It follows from (1) that

(6) $\{T', T_0(a_1, ..., a_n) \supset E\} \vdash E$,

where $T_0(a_1, ..., a_n)$ is the result of replacing the distinct variables in the quantifier-free part of T by distinct individual constants not occurring in E. The hypothesis of the theorem insures that

(7) $T_0(a_1, ..., a_n) \supset E$ is singular and true.

Since T is not logically true, neither is $T_0(a_1, ..., a_n)$. Thus, by the argument establishing (5) in the proof of Theorem 1, there is a consistent class K of basic sentences such that

(8) $K \vdash \sim T_0(a_1, ..., a_n)$,

and the sentences in K contain no predicates or individual constants beyond those in $T_0(a_1, ..., a_n)$. Therefore, by Lemma 4,

(9) E is not logically derivable from K,

and $\sim E$ is also not logically derivable from K. Hence, by (1),

(10) $\sim T'$ is not logically derivable from K.

It follows from (8) that

(11) $K \vdash T_0(a_1, ..., a_n) \supset E$.

Thus, by (3), (4), (7), (6), (11), (9), and (10), $(T', T_0(a_1, ..., a_n) \supset E)$ is an explanans for E. The theorem now follows, using (3) and (2).

Theorem 4. Let T be a fundamental theory such that some singular sentence is explainable by T, and E a true singular sentence which is not logically provable. Assume in addition that L contains infinitely many individual constants. Then there is a derivative law logically derivable from T by which E is explainable.

Proof. Let E_0 be a singular sentence explainable by T. Then there will be a singular sentence $C \supset E_0$ which is logically derivable from T but not logically provable. Clearly there is a universal generalization T_0 of $C \supset E_0$ which is a fundamental law, logically derivable from T, and not logically provable. Applying Theorem 3 to T_0 and E, we obtain a derivative law satisfying the conclusion of the present theorem.

We now state an analogue to Theorem 4; whereas the latter asserts the existence of an intermediate *theory* between almost any T and E, the next theorem asserts the existence of an intermediate singular sentence.

Theorem 5. If T is a fundamental theory such that some singular sentence is explainable by T, and E is a true singular sentence which is not logically provable, then there is a singular sentence explainable by T from which E is logically derivable.

Proof. As in the proof of Theorem 4, we can find a singular sentence S which is logically derivable from T and not logically provable. Further, S can be chosen so that it has no individual constants in common with E. Let C be a non-provable true disjunct of a disjunctive normal form of E. Since $\vdash C \supset E$, it follows that $\vdash (S \cdot C) \supset E$. To complete the proof, it is sufficient to show that (T, C) is an explanans for $S \cdot C$. Clauses (1), (3), and (4) of the definition are clearly satisfied, and clause (2) follows from the fact that some singular sentence is explainable by T. Let K be the class of basic conjuncts of C. Then K is a class of basic sentences from which C is logically derivable. $\sim T$ is certainly not derivable from K, since all elements of K are true and $\sim T$ is false. It remains only to show that $S \cdot C$ is not logically derivable from K. But if it were, we could conclude that $K \vdash S$, which with Lemma 4 yields either $\vdash \sim C$ or $\vdash S$. This, however, is impossible in view of the truth of C and the non-provability of S.

Our last two theorems are not intended as trivializations, but have to do with the replacement, for purposes of explanation, of theories by laws.

Theorem 6. If T is a fundamental theory and E is a singular sentence explainable by T, then there is a fundamental law which is logically derivable from T and by which E is explainable.

Proof. From the hypothesis of the theorem we see that there is a true singular sentence C and a class K of basic sentences such that,

(1) $\vdash T \supset (C \supset E)$,
(2) $K \vdash C$, and
(3) neither $\sim T$ nor E is logically derivable from K.

Let C and E have the respective forms

$$C_0(a_1, \ldots, a_n),$$
$$E_0(a_{n+1}, \ldots, a_m),$$

where the individual constants $a_1, \ldots, a_m$ are not necessarily distinct. Let L be the sentence

$$(x_1), \ldots, (x_m) [C_0(x_1, \ldots, x_n) \supset E_0(x_{n+1}, \ldots, x_m)],$$

where $C_0(x_1, \ldots, x_n)$ and $E_0(x_{n+1}, \ldots, x_m)$ are the results of replacing each individual constant a_i in C and E by the variable x_i, subject to the condition that $x_i = x_j$ if and only if $a_i = a_j$. By (1) and Lemma 3,

(4) $\vdash T \supset L$.

Now L is a true sentence consisting of universal quantifiers followed by an expression without quantifiers or individual constants. Hence

(5) L is a fundamental law.

We must show that

(6) L is not logically equivalent to any singular sentence.

Assume to the contrary that there is a singular sentence S such that,

(7) $\vdash S \equiv L$.

Then by Lemma 1 we have:

(8) either $\vdash \sim S$ or $\vdash L$.

But (8) is impossible, since the first disjunct of (8), with (7), implies that $\vdash \sim L$ in contradiction to the truth of L; and the second disjunct of (8) implies that $\vdash C \supset E$, contrary to (2) and (3). Therefore (6) is established.

From (3) and (4) we have:

neither $\sim L$ nor E is logically derivable from K.

Hence, by (5), (6), (2), and the obvious fact that $\{L, C\} \vdash E$, we conclude that

E is explainable by L,

which completes the proof.

Theorem 7. Let T be a derivative theory and E a singular sentence explainable by T. Assume in addition that L contains infinitely many one-place predicates. Then there is a derivative law which is logically derivable from T and by which E is explainable.

Proof. Assume that T is a derivative theory and E a singular sentence explainable by T. Then there is a true singular sentence C and a class K of basic sentences such that

(1) $\{T, C\} \vdash E$,
(2) $K \vdash C$,
(3) not $K \vdash E$,
(4) not $K \vdash \sim T$.

We may clearly impose a further requirement:

No predicate occurs in members of K which does not occur in C. Let F be a one-place predicate which does not occur in T, C, or E, and let a be any individual constant. Either Fa is true or $\sim Fa$ is true. Assume, without loss of generality,

(5) Fa is true.

Let L be the sentence,

$$(x)[(Fx \cdot C) \supset E].$$

We claim that L is the required derivative law. To show that,

(6) L is a derivative law,

it suffices to establish that,

(7) L is logically derivable from a fundamental law,

and

(8) L is not logically equivalent to a singular sentence.

Since T is a derivative theory, there is some fundamental theory T' such that $\{T'\} \vdash T$. Thus, by (1), $\{T'\} \vdash L$. We may now apply Lemma 3 to conclude that $\{T'\} \vdash L'$, where L' is the universal generalization of the result of replacing the distinct individual constants of L by distinct variables. L' is true since it is logically derivable from the true sentence T'; hence L' is a fundamental law, and (7) follows.

To establish (8), assume to the contrary that there is a singular sentence S such that $\vdash S \equiv (x)((Fx \cdot C) \supset E)$. Hence, since C and E are singular,

(9) $\vdash S \equiv ((C \cdot \sim E) \supset (x) \sim Fx)$.

Therefore $\vdash (S \cdot C \cdot \sim E) \supset (x) \sim Fx$, which, by Lemma 1, implies that either $\vdash \sim (S \cdot C \cdot \sim E)$ or $\vdash (x) \sim Fx$. Since, by (5), $(x) \sim Fx$ is false, it follows that $\vdash \sim (S \cdot C \cdot \sim E)$, or, equivalently,

$$\vdash S \supset (C \supset E).$$

Thus, by (9),

$$\vdash (x) \sim Fx \supset (C \supset E).$$

Recalling that F does not occur in $C \supset E$, we infer, by Lemma 2, that

either $\vdash \sim (x) \sim Fx$ or $\vdash C \supset E$.

But the first alternative clearly fails and the second alternative contradicts (2) and (3); thus we have established (8), which with (7) completes the proof of (6).

By (1),

(10) $\{T\} \vdash L$.

To show that E is explainable by L, we form C' and K' as follows.

$C' : Fa \cdot C$

$K' : K \cup \{Fa\}$

By (5),

(11) C' is singular and true.

By the construction of L, C', and K',

(12) $\{L, C'\} \vdash E$,

(13) K' is a class of basic sentences.

Using (2),

(14) $K' \vdash C'$.

To show that

(15) E is not logically derivable from K',

assume the contrary. Then

$K \vdash Fa \supset E$,

and hence, by the Substitution Rule on Predicates and the fact that F occurs in neither E nor the members of K, $K \vdash (Fa \vee \sim Fa) \supset E$, which contradicts (3).

It remains only to show that

(16) $\sim L$ is not logically derivable from K'.

Assume to the contrary that

$\{(x)[(Fx \cdot C) \supset E]\} \cup K \vdash \sim Fa$.

It follows that

$K \vdash Fa \supset \sim E$,

and hence, again using the Substitution Rule on Predicates, that $K \vdash \sim E$. Therefore, by (1) and (2), $K \vdash \sim T$, which contradicts (4).

The conclusion of the theorem follows from (6), (10), (8), (11), (12), (13), (14), (15), and (16).

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