Finitude and Infinitude in the Atomic Calculus of Individuals

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Nelson Goodman has asked[1] whether there is any sentence in the language of his calculus of individuals which says whether there are finitely or infinitely many atoms. More precisely: is there any sentence that is true (false) in every finite intended model, regardless of its size, but not in any infinite atomic intended model? We shall show that there is no such sentence. We cannot say that there are finitely (infinitely) many atoms unless 1) we say something more specific about the number of atoms, or 2) we enlarge the language by providing for infinite conjunctions or disjunctions, or 3) we enlarge the language by providing suitable new predicates.

The language of the calculus of individuals is first-order logic with the following vocabulary of predicates:

$x \circ y$ (x overlaps y)	Sxyz (the sum of x and y is z)
$x < y$ (x is part of y)	Nxz (the negate of x is z)
$x = y$ (x is identical with y)	Ax (x is an atom)

The *definitional axioms* of the calculus of individuals are (the universal closures of):

D1. $x<y \equiv \forall z(z \circ x \supset z \circ y)$
D2. $x=y \equiv \forall z(z \circ x \equiv z \circ y)$
D3. $Sxyz \equiv \forall w(w \circ z \equiv w \circ x \vee w \circ y)$
D4. $Nxz \equiv \forall w(w<z \equiv \sim w \circ x)$
D5. $Ax \equiv \forall z(z<x \supset z=x)$

[1] In lectures on Goodman, *The Structure of Appearance* (Cambridge, Mass.: Harvard University Press, 1951).

The *existential axioms* of the calculus of individuals are (the universal closures of):

$$\text{E1. } \exists z(z<x \,\&\, z<y) \equiv xoy$$
$$\text{E2. } \exists z(Sxyz)$$
$$\text{E3. } \exists z(Nxz) \equiv \forall \sim w(wox)$$

The *axiom of atomicity* is:

$$\text{AA. } \forall x \exists y(Ay \,\&\, y<x)$$

The *atomic calculus of individuals* (henceforth ACI) is the theory axiomatized by D1-D5, E1-E3, and AA. By examining Goodman's discussion of the calculus of individuals,[2] it is easy to verify that the definitional and existential axioms hold in every intended model. The axiom of atomicity holds in just those intended models—*atomic* intended models—in which everything consists entirely of atoms. We are not concerned with the remaining intended models, in which things may consist wholly or partly of infinitely divisible nonatomic stuff. All finite intended models are atomic.

For any positive number n, we can write a certain sentence saying that there are at least n atoms. (Later we will say exactly which such sentence it is to be.) Call these sentences *numerative sentences*. We shall prove the following *Normal Form Theorem*:

> *Any sentence in the language of ACI is equivalent in ACI to a truth-functional compound of numerative sentences, and there is an effective procedure for finding one such equivalent of any given sentence.*

Given this theorem, our negative answer to Goodman's question is an easy corollary. Call a sentence *indiscriminate* if and only if it has the same truth value in every infinite atomic intended model and also in every finite intended model with sufficiently many atoms. We seek a sentence that is not indiscriminate, being true in every finite intended model but false in every infinite atomic intended model (or vice versa). But every numerative sentence is indiscriminate, being true in every finite or infinite intended model with more than some number of atoms. And since every negation of an indiscriminate sentence is indiscriminate, and every conjunction of indiscriminate sentences is indiscriminate, every truth-func-

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[2] *Structure of Appearance*, II, 4. We have adapted Goodman's treatment only by introducing the predicates S and N; Goodman uses the corresponding functors, defined by means of definite descriptions.

tional compound of numerative sentences is indiscriminate. Every sentence equivalent in ACI to an indiscriminate sentence is indiscriminate. Therefore every sentence is indiscriminate, and the sentence we seek does not exist.

The Normal Form Theorem has several other interesting consequences. For any positive number n, let ACI_n be the theory obtained from ACI by adding as further axioms the numerative sentence saying that there are at least n atoms and the negation of the numerative sentence saying that there are at least $n+1$ atoms; and let ACI_∞ be the theory obtained from ACI by adding as further axioms every numerative sentence. Call these theories *numerative extensions* of ACI. Assuming that intended models come in all finite, and some infinite, sizes—*size* being number of atoms—each numerative extension of ACI is the theory of a nonempty class of intended models: ACI_n is the theory of all intended atomic models of size n, ACI_∞ is the theory of all infinite atomic intended models. (We can restate our negative answer to Goodman's question thus: ACI_∞ is not finitely axiomatizable.) In each numerative extension of ACI, every numerative sentence can be effectively proved or disproved. Therefore the Normal Form Theorem provides a decision procedure for each numerative extension of ACI. It follows that the numerative extensions of ACI are maximal consistent, and it is easy to show that they are the only maximal consistent extensions of ACI. It also follows that ACI is a semantically complete theory of the class of all atomic intended models: given any set of sentences consistent with ACI, it can be embedded in a maximal consistent extension of ACI which, by our previous result, is one of the numerative extensions of ACI and therefore is true in the atomic intended models of the appropriate size. Finally, provided we disregard those models in which identity receives a nonstandard interpretation,[3] it follows that any finite model of ACI, intended or not, is isomorphic to an intended model: it satisfies some ACI_n along with the intended models of size n, and any two finite models of the same maximal consistent theory with (standardly interpreted) identity are isomorphic. We have obtained these results about atomic intended models without ever saying what those are; it was sufficient to know that they are models of ACI, they are standard

[3] That would be taken for granted if we regarded $=$ as a logical constant, as is customary. But, following Goodman (*Structure of Appearance,* II, 4, D2.044), we do not, but rather adopt as a nonlogical axiom D2, in which it is defined in terms of the nonlogical predicate **o**.

with respect to identity, and they come in all finite, and some infinite, sizes.

It only remains to prove the Normal Form Theorem. We obtain it as a special case of a stronger normal form theorem, applicable not only to sentences but to all formulas in the language of ACI.

If n is any positive number and X and Y are any disjoint finite sets of variables, let Z be the set of the alphabetically first n variables not in X or Y, and let $[n, X, Y]$ be the formula which is the existential closure with respect to all the variables in Z (in alphabetical order) of the conjunction (in alphabetical order) of all the following formulas:

1) every formula $A\alpha$ with α in Z;
2) every formula $\sim\alpha=\beta$ with α in Z and β in $Z - \{\alpha\}$;
3) every formula $\alpha \mathbf{o} \beta$ with α in Z and β in X;
4) every formula $\sim\alpha \mathbf{o} \beta$ with α in Z and β in Y.

Let $W = X \cup Y$, so that W is the set of variables occurring free in $[n, X, Y]$; then we call $[n, X, Y]$ a W-*numerative formula*. Now we can define a *numerative sentence* as a sentence $[n, \wedge, \wedge]$ for some n, where $\wedge$ is the empty set; that is, as a $\wedge$-numerative formula. Call a formula ϕ *normalizable* if and only if ϕ is effectively equivalent in ACI to a truth-functional compound of W-numerative formulas, where W is the set of variables free in ϕ. Our Normal Form Theorem, which says that every sentence is normalizable, follows from the theorem:

Every formula is normalizable.

Proof: It is sufficient to prove that any formula whose only predicate is $\mathbf{o}$ is normalizable, since every formula is effectively equivalent in ACI to such a formula, by the definitional axioms of ACI. Let ϕ be any such formula with the set W of free variables. We prove that ϕ is normalizable by induction on the complexity of ϕ.

Case 1: ϕ is $\alpha \mathbf{o} \beta$. Then ϕ is normalizable, being equivalent in ACI to $[1, \{\alpha, \beta\}, \wedge]$.

Case 2: ϕ is a truth-functional compound of normalizable formulas. Let ψ^i be any one of these; by hypothesis, it is effectively equivalent in ACI to a truth-functional compound of V^i-numerative formulas with V^i a subset of W. But any V^i-numerative formula is effectively equivalent to a truth-functional compound of W-numerative formulas; add the variables in $W - V^i$ one at a time by repeated use of the logical equivalence of any numerative formula $[n, X, Y]$ to the disjunction of all the following formulas, where α is the alphabetically first variable not in $X \cup Y$:

1) $[n, X \cup \{\alpha\}, Y]$;
2) $[n, X, Y \cup \{\alpha\}]$;
3) every conjunction $[m, X \cup \{\alpha\}, Y]$ & $[n\text{-}m, X, Y \cup \{\alpha\}]$ with $0 < m < n$.

Case 3. ϕ is an existential or universal quantification of a normalizable formula. The case of a universal quantifier reduces to that of an existential quantifier in view of case 2; so assume that ϕ is $\exists\, \alpha\psi$, with ψ normalizable. It can be shown that ψ is effectively equivalent to a disjunction $\chi^1 \vee \ldots \vee \chi^k$, in which each formula χ^i is a conjunction of formulas χ^i_V indexed by all subsets of V of W and each formula χ^i_V is the conjunction of a formula of form 1, 2, or 3 and a formula of form 4, 5, or 6.

1) $\sim [m + 1, V \cup \{\alpha\}, W - V]$ with $m = 0$
2) $[m, V \cup \{\alpha\}, W - V]$ & $\sim [m + 1, \cup \{\alpha\}, W - V]$
3) $[m, V \cup \{\alpha\}, W - V]$
4) $\sim [n + 1, V, (W - V) \cup \{\alpha\}]$ with $n = 0$
5) $[n, V, (W - V) \cup \{\alpha\}]$ & $\sim [n + 1, V, (W - V) \cup \{\alpha\}]$
6) $[n, V, (W - V) \cup \{\alpha\}]$

Intuitively, each assignment of values to the variables in W partitions the atoms into cells C_V indexed by all subsets V of W: C_V contains just those atoms which overlap the value of every variable in V but not the value of any variable in $W - V \cdot \chi^i_V$ says that C_V contains exactly (or at least) m atoms of α and exactly (or at least) n other atoms, with $m, n \geqslant 0$. It is sufficient to show that each $\exists\, \alpha\chi^i$ is normalizable, since ϕ is plainly equivalent to the disjunction of these. Consider two subcases.

Case 3a: For every subset V of W, the first conjunct of χ^i_V has the form 1. Then $\exists\, \alpha\chi^i$ is inconsistent with ACI, and hence is equivalent in ACI to the conjunction of every formula $\sim [1, V, W - V]$ with V a subset of W.

Case 3b: Otherwise. Then $\exists\, \alpha\chi^i_V$ is equivalent in ACI to the conjunction of the formulas $\exists\, \alpha\chi^i_V$. It is sufficient to show that each of these is normalizable; consider any one. If the conjuncts of χ^i_V have the forms 1 and 4, $\exists\, \alpha\chi^i_V$ is equivalent in ACI to $\sim [1, V, W - V]$. If the conjuncts of χ^i_V have the forms 1 and 5, 2 and 4, or

2 and 5, $\exists\, \alpha\chi^{i}_{V}$ is equivalent in ACI to $[m + n,\ V, W - V]$ & $\sim$ $[m + n, + 1, V, W - V]$. Otherwise $\exists\, \alpha\chi^{i}_{V}$ is equivalent in ACI to $[m + n, V, W - V]$.

This completes the proof.[4]

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[4] The authors thank David Kaplan for some helpful remarks.